Faithful interpretations of **LJT** and **LJQ** into polarized logic

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Summary

- **LJT** and **LJQ** are well-known intuitionistic focused sequent calculi, connected with call-by-name and call-by-value computation.
- We revisit polarized logic as a unifying framework for the simultaneous study of proof search in **LJT** and **LJQ**, developing faithful embeddings for provability and proofs.
- Then, we can employ the tools of coinductive proof search for polarized logic to address inhabitation questions in **LJT** and **LJQ**.
- The novelty in the talk comes from the development of the idea for **LJQ** (the treatment for **LJT** is found in TYPES'2020 post-proc.).

Part I

Background on coinductive proof search and polarized logic

Coinductive approach to proof search

- Extend the Curry-Howard paradigm (of representation of proofs by typed lambda-terms) to solutions of proof search problems:
  - a solution is a run of the proof search process that does not fail to apply bottom-up an inference rule, so it may be an infinite object;

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  - e.g., in normal intuitionistic natural deduction / STLC, sequent

$$\sigma := (f : a \supset a, x : a) \Rightarrow a \qquad (a \text{ an atom})$$

has (inf. many) finite solutions, but also one infinite solution $f\langle f\langle ...\rangle\rangle$.

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  - the other inductive, enriching proof terms with a formal fixed-point operator to represent cyclic behaviour;

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  - both employing formal finite sums to represent choice points, and entire solution spaces;
  - e.g., the solution space of the type of *Church numerals* $(a \supset a) \supset a \supset a$ gets the finitary representation
$$\lambda f^{a \supset a}. \lambda x^a. \mathsf{gfp}\, Y^\sigma . (f\langle Y^\sigma \rangle + x) \qquad (\sigma \text{ as above}).$$

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Formulas:

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\begin{array}{rrcl}
 & A & ::= & N \mid P \\
\text{(negative)} & N, M & ::= & a^{-} \mid C \\
\text{(composite negative)} & C & ::= & P \supset N \mid N \wedge M \mid \uparrow P \\
\text{(positive)} & P, Q & ::= & a^{+} \mid \bot \mid P \vee Q \mid \downarrow N
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Auxiliary classes:

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Contexts Γ: sets of declarations $(x : L)$ (only left formulas)

Focused sequent calculus **LJP** (cont.)

Sequent forms:

$\Gamma \Longrightarrow t : N$	(invert negative right)
$\Gamma \mid p : P \Longrightarrow A$	(invert positive left)
$\Gamma[s : N] \vdash R$	(focus negative left)
$\Gamma \vdash [v : P]$	(focus positive right)
$\Gamma \vdash e : A$	(stable)

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2. such choices place one formula in focus with one of the rules:

$$\frac{\Gamma \vdash [v : P]}{\Gamma \vdash \mathrm{ret}(v) : P} \qquad \frac{\Gamma, x : N\,[s : N] \vdash R}{\Gamma, x : N \vdash \mathrm{coret}(x, s) : R} \qquad (\text{recall } R ::= P \mid a^-)$$

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3. the formula in focus is then successively decomposed (by a focusing stage, followed by an inversion stage), until stable sequents emerge again (or axioms or "dead-ends" reached).

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  - σ is inhabited in **LJP** iff EF($\mathcal{F}(\sigma)$);
  - σ has finitely many inhabitants in **LJP** iff FF($\mathcal{F}(\sigma)$).

Part II: Faithful embeddings of **LJQ** and **LJT** into polarized logic (with application to inhabitation questions)

Intuitionistic focused sequent calculus **LJQ**

Cut-free fragment of **LJQ** (after a presentation by Dyckhoff-Lengrand):

forms of sequents:	$\Gamma \vdash t : A$	(term sequents – stable)
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typing rules (implication-disjunction fragment):

$$\frac{\Gamma, x : A \supset B \vdash [v : A] \quad \Gamma, x : A \supset B, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash x(v, y^B.t) : C}$$

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$$\frac{\Gamma, x : A_1 \vee A_2, y_i : A_i \vdash t_i : B \quad \text{for } i = 1, 2}{\Gamma, x : A_1 \vee A_2 \vdash x(y_1^{A_1}.t_1, y_2^{A_2}.t_2) : B} \qquad \frac{\Gamma \vdash [v : A]}{\Gamma \vdash \langle v \rangle : A}$$

$$\frac{}{\Gamma, x : a \vdash [x : a]} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash [\lambda x^A.t : A \supset B]} \qquad \frac{\Gamma \vdash [v : A_i]}{\Gamma \vdash [\mathrm{in}_i(v) : A_1 \vee A_2]}$$

Faithful embedding of **LJQ** into polarized logic

We define an embedding $(\_)^{\mathfrak{p}} : \textbf{LJQ} \longrightarrow \textbf{LJP}$, simultaneously with $(\_)^{\bar{\mathfrak{p}}}$, both producing positive formulas of **LJP**, applied to positive (resp. negative) suformula occurrences of the **LJQ** formula at hand:

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Soundness: $\dfrac{\Gamma \vdash [v : A]}{\Gamma^{\mathfrak{l}} \vdash [v^{\mathfrak{p}} : A^{\mathfrak{p}}]}$ $\qquad$ $\dfrac{\Gamma \vdash t : A}{\Gamma^{\mathfrak{l}} \vdash t^{\mathfrak{p}} : A^{\mathfrak{p}}}$

with $(\_)^{\mathfrak{l}}$ producing left formulas of **LJP**: $a^{\mathfrak{l}} = a^{+}$ and $A^{\mathfrak{l}} = N$ if $A^{\bar{\mathfrak{p}}} = \downarrow N$.

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Faithfulness (with the help of a forgetful map $|\_|$ on the image of $(\_)^{\mathfrak{p}}$): $\Gamma^{\mathfrak{l}} \vdash e : A^{\mathfrak{p}}$ in **LJP** implies e in the image of $(\_)^{\mathfrak{p}}$ and $\Gamma \vdash |e| : A$ in **LJQ**.

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In fact, the forgetful map is inverse to $(\_)^{\mathfrak{p}}$, and gives a bijection of inhabitants of σ in **LJQ** and of $\sigma^{\mathfrak{p}}$ in **LJP**.

Deciding inhabitation problems in **LJQ** and **LJT**

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a^* & = & a^- & & & & \\
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- In **LJQ** trivial unfoldings may appear just with implication.

  For example, infinitely many inhabitants are possible for

  $$x : a \supset b, y : a \vdash b$$

  (x and y can be used to produce arbitrarily many copies $z_i : b$).

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